From (2) $d\phi=\cos\phi d\theta/\tan\beta$......(3).
(3) in (1) gives

$$ds=r\cos\phi\sqrt{(1+\tan^2\beta)}d\theta/\tan\beta=r\cos\phi d\theta/\sin\beta.$$

From (2), $\theta=\tan\beta\int\frac{d\phi}{\cos\phi}=\tan\beta\log[\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)]$.

$\therefore\ e^{\theta\cot\beta}=\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)=\frac{1+\sin\phi}{\cos\phi}$. $\therefore\ \cos\phi=\frac{2}{e^{\theta\cot\beta}+e^{-\theta\cot\beta}}=\frac{2}{e^{\theta}+e^{-\theta}}$,

since $\cot\beta=1$.

$$\therefore\ s=2r\sqrt{2}\int_0^{\theta}\frac{d\theta}{e^{\theta}+e^{-\theta}}=2r\sqrt{2}(\tan^{-1}e^{\theta}-\tfrac{1}{4}\pi).$$

When $\theta=\frac{1}{2}\pi$, $s=2r\sqrt{2}(\tan^{-1}e^{\frac{1}{2}\pi}-\frac{1}{4}\pi)=r\sqrt{2}(.3695185\pi)=6494.764423$ miles.
When $\theta=\pi$, $s=2r\sqrt{2}(\tan^{-1}e^{\pi}-\frac{1}{4}\pi)=r\sqrt{2}(.472506\pi)=8304.902620$ miles.
When $\theta=\frac{3}{2}\pi$, $s=2r\sqrt{2}(\tan^{-1}e^{\frac{3}{2}\pi}-\frac{1}{4}\pi)=r\sqrt{2}(.494281\pi)=8687.626341$ miles.
When $\theta=2\pi$, $s=2r\sqrt{2}(\tan^{-1}e^{2\pi}-\frac{1}{4}\pi)=r\sqrt{2}(.498804\pi)=8767.1239018$ miles.

Also solved by *J. SCHEFFER*. A somewhat different solution of this problem is given in Finkel's *Mathematical Solution Book*, page 344.

96. Proposed by W. H. CARTER, Vice President, and Professor of Mathemetics, Centenary College, Jackson, La.

If $f(x)=\int f(x)dx$, find $f(x)$, the constant being zero.

Solution by W. F. SHAW, 1600 Sabine Street, Austin, Tex.

$f(x)=\int f(x)dx$. $df(x)=f(x)dx$.

$\frac{df(x)}{f(x)}=dx$. $\log f(x)=x$. $f(x)=e^x$.

Also solved by *W. H. DRANE*, *J. SCHEFFER*, and *G. B. M. ZERR*.

MECHANICS.

87. Proposed by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

"He on his impious foes onward drove,
Drove them before him to the bounds
And crystal walls of Heaven; which opening wide
Rolled inward and a spacious gap disclosed
Into the wasteful deep; headlong themselves they threw
Down from the verge of Heaven.
Nine days they fell; Hell at last
Yawning received them whole and on them closed!"
Paradise Lost, Book VI.

Assuming Hell to be the center of the earth, and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

Solution by P. H. PHILBRICK, C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.

Let a be the distance from the center of the earth, or origin. g the force of gravity at the surface. Then the equation outside the earth is, $dx^2/dt^2=-gr^2/x^2$. Multiply by $2dx$, integrate, and we find,

$$\frac{dx^2}{dt^2}=v^2=2gr^2\left(\frac{1}{x}-\frac{1}{a}\right)\ldots\ldots(1).$$

At the surface of the earth $x=r$, and we have from (1),

$$v=2gr\left(\frac{a-r}{a}\right)^{\frac{1}{2}}=\sqrt{2gr}, \text{ very nearly}, \quad \ldots\ldots(2),$$

since a is very great compared with r.

The force of gravity inside the earth and distant x from the center is, $g(x/r)=(g/r)x$.

Hence the equation of motion is, $d^2x/dt^2=-(g/r)x$.

Integrating we have, $dx^2/dt^2=(g/r)(r^2-x^2)$, since the "lost spirits," so far as this force is concerned, are supposed at rest when $x=r$ and $t=0$.

From above,

$$\sqrt{\frac{g}{r}}\,dt=\frac{-dx}{\sqrt{(r^2-x^2)}}. \quad \therefore\ t=\sqrt{\frac{r}{g}}\cos^{-1}\frac{x}{r}\ldots\ldots(3).$$

From (2) we have

$$\frac{r}{v}=\sqrt{\frac{r}{2g}}\left(\frac{a}{a-r}\right)^{\frac{1}{2}}=\sqrt{\frac{r}{2g}}, \text{ very nearly}=t'\ldots\ldots(4).$$

This gives the time of passing from the surface to the center of the earth by virtue of the velocity at the surface.

Let $r=20880000$ and $g=32$, and (3) and (4) become,

$$t=808\cos^{-1}\frac{x}{r}\ \ldots\ (3'), \text{ and } t'=\sqrt{\frac{r}{2g}}=571.2\ \ldots\ (4').$$

We may find from these equations by trial, the time required for the spirits to pass from the surface to the center of the earth.

Thus in (3′) let $x=.83195a$.

Then $t=808\cos^{-1}.83195=808\times.588176=475.246$, and $t'=.83195\times571.2=475.21$.

About $\frac{5}{6}$ of the radius of the earth is traversed by virtue of the velocity at the surface, and the remaining $\frac{1}{6}$ by virtue of the interior attractive force.

$$\text{Nine days} = 9\times24\times60\times60 = 777600 \text{ seconds}$$
$$\text{Deduct} \quad 475 \text{ seconds}$$

$\therefore$ Time of falling to surface of the earth=777125 seconds.

The corresponding distance is, $\frac{1}{2}gt^2=16\frac{1}{12}(777125)^2=9713099188802$ feet $=1839602119$ miles.

Also solved by *H. C. WHITAKER*, *G. B. M. ZERR*, *B. F. SINE*, *J. SCHEFFER*, *W. H. DRANE*, and *J. B. GREGG*. Mr. Gregg furnished a very elaborate solution, indicating the various steps of the computation, his result being 360733 miles. Dr. Whitaker gets the same result. The results of the various contributors differ widely, due to variously assumed values of the constants, and, in some cases, considering the earth a mere point.

95. Proposed by FLORIAN CAJORI, Ph. D., Author of History of Mathematics, History of Physics, etc., and Professor of Mathematics, Colorado College, Colorado Springs, Col.

Assuming that the velocity is proportionate to the distance described from the state of rest, (1) can the body start in motion? (2) If it can, what is its initial acceleration? If we make the additional assumption that the time of fall, from rest, through a finite distance is finite, does it follow that the distance is infinite?

Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

1. The statement here is somewhat confusing. No body can start in motion unless acted upon by an external force; if it be meant here then to ask, can the body be started, the answer is self-evident. Theoretically, an infinitesimal force would be sufficient to put the body in motion though the time might become infinite before the velocity became finite.

2. We have $ds/dt=ks$ where k is a constant depending upon initial conditions. Differentiating we get $d^2s/dt^2=k(ds/dt)=k^2s$; that is, the initial acceleration is k time the initial velocity and is itself proportional to the distance. What the value of this initial acceleration is, depends upon the external force acting and the mass of the body.

3. Nothing is said here about the initial force acting. If we assume it infinitesimal, it follows from 1, that if the time of fall through a finite distance is finite, then the velocity must be infinite.

DIOPHANTINE ANALYSIS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics in University of Tennessee, Knoxville, Tenn.

Find three square numbers in harmonical progression.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The terms of an harmonical progression are the reciprocals of such numbers as form an arithmetical progression.

Let a^2, b^2, and c^2 be three square numbers in arithmetical progression, $a<b<c$. Then $b^2-a^2=c^2-b^2$, or $a^2+c^2=2b^2$. a, b, and c are rendered rational and integral by putting $a=p^2-q^2$ the difference between $2pq$, $b=p^2+q^2$, and $c=p^2-q^2+2pq$.